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Permutation orientifolds

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Abstract: We consider orientifold actions involving the permutation of two identical factor theories. The corresponding crosscap states are constructed in rational conformal field theory. We study group manifolds, in particular the examples $SU(2) \times SU(2)$ and $U(1) \times U(1)$ in detail, comparing conformal field theory results with geometry. We then consider orientifolds of tensor products of $N = 2$ minimal models, which have a description as coset theories in rational conformal field theory and as Landau Ginzburg models. In the Landau Ginzburg language, B-orientifolds and D-branes are described in terms of matrix factorizations of the superpotential. We match the factorizations with the corresponding crosscap states.

Keywords: Conformal Field Models in String Theory, D-branes.

Contents

1. Introduction

Orientifolds [1] play a prominent role in the context of model building. For this and other reasons it is of interest to understand their construction and physical properties in detail.

Given a tensor product of two identical theories, it is a natural idea to consider orientifolds that involve the permutation of the two factors. Such models can then be studied from various points of view. If each single factor is a rational conformal field theory, one can approach the problem algebraically and construct a crosscap state describing a parity action that combines the permutation with world sheet orientation reversal and possibly other actions. For a single factor, this approach has been employed initially in [2] [1] . On the other hand, the problem can also be approached geometrically. Given a sigma-model with target space $M \times M$ one can consider an involution exchanging the two factors, the fixed point set being the diagonally embedded $M \subset M \times M$.

In this paper, we will be interested in models that can be studied from both points of view.

We start out setting the stage in rational conformal field theory, giving an explicit construction of permutation crosscap states. One motivation for the form of the crosscap

[1] A novel approach in the context of rational conformal field theory is taken in [3], where a full construction of the correlators on orientable and non-orientable world sheets is given. We will not employ it in the current paper.

state comes from the idea that geometrically the orientifold should be localized on the diagonal of the two factors. The parity should act trivially on the corresponding D-brane, whose world volume coincides with the orientifold fixed point set. This is enough to determine the crosscap state by a modular transformation to the closed string channel provided that the boundary state is known. We also point out that the form of our crosscap state suggests a generalization of the arguments of [4, 5] for boundary states corresponding to automorphism twisted boundary conditions to the orientifold case. Necessary conditions for the consistency of our proposed crosscap states are provided by the one-loop diagrams, which we summarize in the appendix.

In section 3 we consider orientifolds of products of group manifolds $G \times G$. Even though these manifolds do not appear directly as part of a string model, they are simple models for studying the behavior of D-branes and orientifold planes. They also provide the basics for coset constructions such as those appearing in Gepner models.

The most symmetric D-branes preserving the full underlying symmetry were constructed as rational boundary states by Cardy [6]. The geometry of these D-branes was uncovered in [7, 8] where it was shown that Cardy's boundary states correspond geometrically to D-branes wrapping conjugacy classes of the group manifold. Additional D-brane states preserving an automorphism twisted symmetry were constructed in [5] and shown to wrap twisted conjugacy classes [8]. The non-commutative gauge theory living on their world volume was further investigated in [9]. Permutation branes [10, 11], where the automorphism is the exchange of the symmetry algebras of the two factors, are a special class of such branes, see also the generalization discussed in [12, 13].

Repeating a similar program for unoriented strings, one can relate algebraic and geometric constructions of orientifolds. The crosscap states preserving the full symmetry algebra are those of [2]. Their geometric interpretation on group manifolds has been investigated in [14 − 16] with the result that the orientifold planes are fixed point sets under an involution of the group manifold and are localized on specific conjugacy classes. The basic example is provided by the involution $g \to g^{-1}$, leading to orientifold points sitting at elements of the center of the group whose conjugacy class consists only of one point. This parity action can be modified by multiplication by an element c of the center $g \to cg^{-1}$. In CFT terms, this choice amounts to dressing the parity action by a simple current [17 − 19].

In the classical limit, the closed and open string states have an interpretation in terms of functions on the group manifold (closed strings) or on the conjugacy classes wrapped by the D-brane (open strings). The parity acts as an involution of the group target and induces an action on those function spaces. This geometrical action has been matched with the one read off from conformal field theory one-loop calculations in [14, 16].

In this paper, we consider products of group manifolds and their orientifold actions. The fixed point set of these orientifolds are twisted conjugacy classes, just like in the case of D-branes. We study the examples $\mathrm{SU}(2)_k \times \mathrm{SU}(2)_k$ and $\mathrm{U}(1)_k \times \mathrm{U}(1)_k$ in detail and check the localization of the orientifold plane by scattering closed string states off it [20, 15, 14, 16, 21]. Furthermore, we consider the action of the geometric parity on the space of functions living on the twisted conjugacy class and compare its action with the one derived with the help of conformal field theory. We find complete agreement in the classical (large k) limit of the conformal field theory.

Having studied both $su(2)$ and $u(1)$ in detail, we move on to the coset $su(2)_k \oplus u(1)_2/u(1)_{k+2}$. This is a GSO projected version of the $N = 2$ supersymmetric minimal model which has an alternative description in terms of a Landau-Ginzburg model with superpotential $W = X^{k+2}$. For $N = 2$ theories, there are two types of orientifolds, namely A-type (those that for sigma-models correspond to fixed point sets of an anti-holomorphic involution) and B-type (corresponding to holomorphic involutions). For a single minimal model, the crosscaps and parities have been studied in [22], and results from the Landau-Ginzburg and conformal field theory analysis, such as the action of the parity on D-branes and open string states, have been found to agree. One of the main applications for supersymmetric minimal models is the construction of exact string vacua via Gepner models, whose orientifolds have been studied in [23].

Concerning the extension to permutation orientifolds, the techniques discussed so far in this paper are directly applicable to the (non spin-aligned) tensor product of two coset models, each with a separate GSO projection. However, one would really be interested in the (GSO projected) tensor product of two spin aligned $N = 2$ minimal models, which is what we consider in section 4, focussing on the B-type case. We observe that the parity action on open and closed string states is to some extend inherited from that of the constituent $su(2)_k$ and $u(1)$ theories. However, there are a number of choices of how the parity can act on the $u(1)_2$ part, describing fermions and spin structures. In particular, for any parity action one can consider a related parity that differs in its action on the D-brane from the initial one by a brane-anti-brane flip. In theories with a geometrical interpretation, one would interpret this as an orientation reversal.

We also consider the Landau-Ginzburg description of the tensor product, which has a superpotential $W = X_1^{k+2} + X_2^{k+2}$. In [24] it was explained how to construct topological B-type crosscap states making use of the matrix factorization techniques of [25, 26]. We apply their techniques to the case of the tensor product of two $N = 2$ minimal models. Following [24] one can in particular derive from the Landau Ginzburg point of view how the parity associated to a factorization acts on the B-type D-branes. The latter are also described by matrix factorizations, following the ideas of [27]. We compare these results with the conformal field theory analysis and identify the conformal field theory description of the parity that is most natural in the Landau Ginzburg model.

We do not consider permutation orientifolds of Gepner models, which are however covered in the paper [28].

Note added: some of the results of this paper were independently obtained in the paper [28] by Hosomichi.

2. Permutation boundary and crosscap states

The tensor product of two identical rational conformal field theory models carries a natural action of the permutation group. Accordingly, one can use this symmetry to twist the gluing conditions for boundary and crosscap states.

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We will consider the case of the charge conjugation modular invariant. The symmetry algebra of the model is $\mathcal{A}^1 \otimes \mathcal{A}^2$, where $\mathcal{A}^i = \mathcal{A}^i_L \otimes \mathcal{A}^i_R$, and $\mathcal{A}^i_L = \mathcal{A}^i_R = \mathcal{A}$. The generators of $\mathcal{A}$ are denoted as W. The Hilbert space of the theory is then

$$\mathcal{H} = \bigoplus_{i,j} (\mathcal{H}_i \otimes \mathcal{H}_{\bar{i}}) \otimes \left(\mathcal{H}_j \otimes \mathcal{H}_{\bar{j}}\right) , \tag{2.1}$$

where i, j are representations of $\mathcal{A}$. As usual, one can consider D-branes described by Cardy's boundary states that preserve the full symmetry algebra. The corresponding gluing conditions on the upper half plane are

$$W^{(1)}(z) = \overline{W}^{(1)}(\bar{z}), \quad W^{(2)}(z) = \overline{W}^{(2)}(\bar{z}) \quad \text{for} \quad z = \bar{z} . \tag{2.2}$$

Making use of the permutation symmetry, one can define the following twisted gluing conditions [10]

$$W^{(1)}(z) = \overline{W}^{(2)}(\bar{z}) \quad \text{for} \quad z = \bar{z} \tag{2.3}$$

In the closed string channel, these gluing conditions are implemented on coherent Ishibashi states which fulfill

$$\left(W^{(1)}_n - (-1)^{s_W} \overline{W}^{(2)}_{-n}\right) |\mathcal{B}, (i,i)\rangle\rangle_{(12)} = 0 , \tag{2.4}$$

where the subscript (12) reminds us that this is the Ishibashi state for the permutation boundary conditions. If we let i label the chiral primaries of a single factor, then the permutation Ishibashi states can be built on states in the sector $\mathcal{H}_i \otimes \mathcal{H}_{\bar{i}}$, as in [10]. Explicitly, they are written

$$|\mathcal{B}, (i,i)\rangle\rangle_{(12)} := \sum_{M,N} |i, M\rangle \otimes |i, N\rangle \otimes |\bar{i}, N\rangle \otimes |\bar{i}, M\rangle \tag{2.5}$$

More generally, we can have a pair of indices (i_1, i_2) labeling the tensor product of representations of the symmetry algebra for the left movers, while the summation over M, N runs over the descendants. Permutation Ishibashi states can only be built on ground states whose labels i_1, i_2 agree. The inner product between twisted Ishibashi states is given by

$${}_{(12)}\langle\langle \mathcal{B}, (i,i)| e^{2\pi i \tau H_c} |\mathcal{B}, (j,j)\rangle\rangle_{(12)} = \delta_{ij} \chi_i(2\tau)^2 \tag{2.6}$$

and the one between twisted and untwisted Ishibashi states corresponding to (2.2) by

$${}_{\mathbf{1}}\langle\langle \mathcal{B}, (i_1, i_2)| e^{2\pi i \tau H_c} |\mathcal{B}, (j,j)\rangle\rangle_{(12)} = \delta_{i_1,j} \delta_{i_2,j} \chi_j(4\tau) . \tag{2.7}$$

Permutation boundary states then take the form [10]

$$|\mathcal{B}, J\rangle_{(12)} = \sum_j \frac{S_{Jj}}{S_{0j}} \, |\mathcal{B}, (j,j)\rangle\rangle_{(12)} . \tag{2.8}$$

This is a special case of a more general construction described in [4], taking a novel point of view on [5]. Given an automorphism ω of the chiral algebra, one can consider the ω-twisted boundary conditions, (for the case that ω is a permutation this is (2.4)). One can then

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construct ω twisted Ishibashi states and determine the inner product between two twisted Ishibashi states, which results in characters describing the propagation of strings in the tree-channel. For the case of the permutation automorphism, this is just (2.6). However, in addition there are strings propagating between two branes where the gluing conditions have been twisted by different automorphisms. In particular, one can consider one of the automorphisms to be trivial, leading to Cardy's boundary states, which preserve the full symmetry. The overlap of an ω twisted Ishibashi state and an untwisted Ishibashi state is given by a *twining character*, i.e. a trace over a Hilbert space with an insertion of the induced action of ω on that Hilbert space

$$\chi_\mu(\tau) = \mathrm{Tr}\,_{\mathcal{H}_\mu}\left(\tau_\omega \; e^{2\pi i\tau(L_0-\frac{c}{24})}\right) . \qquad (2.9)$$

In our case, this is just (2.7). Twisted boundary states can be written as linear combinations of twisted Ishibashi states. The coefficients, with which the twisted Ishibashi states are combined to consistent boundary states are restricted by the Cardy condition. To analyze this condition between twisted and untwisted boundary states, one needs to transform the tree level amplitudes to the open string channel, where in particular one makes use of the modular transformation properties of the twining characters [29, 30]. It turns out that one obtains integer combination of suitable twisted characters in the open string channel if one chooses as coefficients the S-matrix elements for the twining characters, divided by the square-root of ordinary S-matrices as usual. The integers appearing in the open string channel are then the twisted fusion rule coefficients, which describe the fusion of a twisted representation with an untwisted one, resulting in a twisted representation, see [4] for details.

In our case, the twining characters appearing in the closed string channel are simply (2.7), and their modular transformation is performed using the ordinary S-matrix of a single model. The coefficients of the permutation boundary state (2.8) are precisely that S- matrix divided by the normalization S_{0j}, following the pattern described above.

Turning to orientifolds, one would similarly like to consider parity actions which involve the exchange of the two symmetry algebras. In the closed string channel, the conditions on the crosscap states can be obtained by conjugating (2.4) with $e^{\pi i L_0}$, where $L_0 = L_0^{(1)} + L_0^{(2)}$. This results in the following condition on crosscap Ishibashi states

$$\left(W_n^{(1)} - (-1)^{s_W+n}\overline{W}_{-n}^{(2)}\right) |\mathcal{C},(i,i)\rangle\rangle_{(12)} = 0 . \qquad (2.10)$$

The crosscap Ishibashi states are obtained from the boundary ones as

$$|\mathcal{C},(i,i)\rangle\rangle_{(12)} = e^{\pi i(L_0-h_i^{tot})}|\mathcal{B},(i,i)\rangle\rangle_{(12)} .$$

The closed string amplitudes between permutation boundary and crosscap states are

$$\begin{aligned}
{}_{(12)}\langle\langle\mathcal{C},(i,i)|e^{2\pi i\tau H_c}|\mathcal{C},(j,j)\rangle\rangle_{(12)} &= \delta_{ij}\chi_i(2\tau)^2 \qquad (2.11)\\
{}_{(12)}\langle\langle\mathcal{B},(i,i)|e^{2\pi i\tau H_c}|\mathcal{C},(j,j)\rangle\rangle_{(12)} &= \delta_{ij}\hat{\chi}_j(2\tau)^2\\
{}_{\mathbf{1}}\langle\langle\mathcal{B},(i_1,i_2)|e^{2\pi i\tau H_c}|\mathcal{C},(j,j)\rangle\rangle_{(12)} &= \delta_{i_1,j}\delta_{i_2,j}\hat{\chi}_j(4\tau) ,
\end{aligned}$$

where $\hat{\chi}(\tau) = e^{-\pi i (h_i - \frac{c}{24})}\chi(\tau + \frac{1}{2})$ as usual. The form of the permutation boundary state suggests the following ansatz for the permutation crosscap state

$$|\mathcal{C}, \mu\rangle_{(12)} = \sum_j \frac{S_{\mu j}}{S_{0j}} |\mathcal{C}, (j,j)\rangle\rangle_{(12)} = \sum_j e^{2\pi i Q_\mu(j)}\ |\mathcal{C}, (j,j)\rangle\rangle_{(12)} \tag{2.12}$$

where μ is a simple current and $Q_\mu(j)$ the monodromy charge of j with respect to μ.

It is interesting to ask if there is an extension of the general construction of twisted boundary states of [5, 4] to the crosscap case. The natural ansatz for a crosscap state would then be to combine the Ishibashi states to full crosscap states using the modular matrix that relates the closed string channel of the *mixed* amplitude between a twisted crosscap state and an untwisted boundary state to the open string channel. It is immediately clear, that the standard PSS crosscaps [2] fall into this pattern: They are given by

$$|\mathcal{C}, \mu\rangle_{\mathbf{1}} = \sum_i \frac{P_{\mu i}}{\sqrt{S_{0i}}} |\mathcal{C}, i\rangle\rangle\,, \tag{2.13}$$

where as before μ is a simple current. The matrix $P = \sqrt{T} S T^2 S \sqrt{T}$ appearing as a prefactor of the Ishibashi state is precisely the one relating open and closed string channel of the untwisted Möbius strip.

What we can seen here is that also the permutation crosscap states obey this construction prescription. Namely, if we consider a mixed amplitude between a permutation crosscap state and a Cardy boundary state, the hatted characters appearing in the closed string channel are transformed to the open string channel using the matrix S for a single factor. The prefactors appearing in the formula for the permutation crosscap state are thus natural from this point of view.

We have listed all one-loop amplitudes involving the permutation crosscap state in appendix A. Our formulas show that the coefficients appearing in the loop amplitudes are integers, thus providing necessary conditions for the consistency of the crosscap state.

Let us highlight a few amplitudes that are of special interest. There is a special permutation D-brane carrying the label $J = 0$, where 0 refers to the vacuum representation. The Cardy brane corresponding to this label would have only a single vacuum character that appears in the open string sector. In the case of the permutation brane, the cylinder amplitude of the $J = 0$ permutation brane with itself takes the form [10]

$${}_{(12)}\langle\mathcal{B}, 0| e^{-\frac{\pi i H_c}{\tau}} |\mathcal{B}, 0\rangle_{(12)} =: \mathcal{C}_{(12)(12)}(0,0) = \sum_j \chi_j(\tau)\chi_{\bar{j}}(\tau) \tag{2.14}$$

and "coincides" with the diagonal bulk partition function if one identifies the right movers with the boundary fields in the second tensor product factor [10]. In the case where a sigma-model interpretation of the theory is available, this brane should describe a brane whose worldvolume is the conjugate diagonal in $M \times M$. We now calculate the Möbius amplitude of this D-brane with the permutation crosscap state

$${}_{(12)}\langle\mathcal{C}, 0| e^{-\frac{\pi i H_c}{4\tau}} |\mathcal{B}, 0\rangle_{(12)} =: \mathcal{M}_{(12)(12)}(0,0) = \sum_j \hat{\chi}_j(\tau)\hat{\chi}_{\bar{j}}(\tau), \tag{2.15}$$

which shows that all open string states are invariant under the orientifold action. We can therefore conclude that our crosscap state corresponds to an orientifold plane that is located on the diagonal. Turning the argument around, we could have postulated that the orientifold we are looking for leaves the brane on the diagonal invariant and acts trivially on all its open string states. By a modular transformation, we could then have concluded that (2.12) is the corresponding crosscap state.
We will use the notation introduced in (2.14) and (2.15) throughout the paper: $\mathcal{C}$, $\mathcal{M}$ and $\mathcal{K}$ stand for the loop channel of the cylinder, Möbius and Klein bottle amplitude respectively. The subscripts refer to the automorphism type of the boundary and crosscap states, whereas the two arguments given in brackets are the Cardy labels of the boundary or crosscap states.

3. Group manifolds

A special example are the cases of group manifolds H [8, 9, 31]. Here, boundary states corresponding to boundary conditions twisted by an automorphism ω have been constructed in [5] and interpreted in terms of branes wrapping twisted conjugacy classes $\mathcal{C}_\omega$ in [8]

$$\mathcal{C}_\omega(g) = \{(h^{-1}g\omega(h)|h \in H\} \,. \tag{3.1}$$

For the special case of products of group manifolds $H = G\times G$ where ω acts as the exchange of the two factors, one obtains

$$\mathcal{C}_{(12)}((g_1, g_2)) = \{(h_1^{-1}g_1h_2, h_2^{-1}g_2h_1) \mid h_i \in G\} \,. \tag{3.2}$$

The brane corresponding to the boundary state $J = 0$ is the twisted conjugacy class of the identity

$$\mathcal{C}_{(12)}((\mathbf{1}, \mathbf{1})) = \{(h, h^{-1})|h \in G\} \tag{3.3}$$

The orientifold actions we want to consider are supposed to exchange the left moving current of the first WZW model with the right moving one of the second. The currents are given by

$$J^{(i)}(z) = kg_i^{-1}\partial g_i \qquad \bar{J}^{(i)}(\bar{z}) = -k\bar{\partial}g_ig_i^{-1}, \qquad i = 1, 2, \tag{3.4}$$

and the possible orientifold actions involve world sheet parity Ω combined with an action on the group manifold $g^{(1)} \to (g^{(2)})^{-1}$. This action can be modified with a translation by an element of the center of the group, corresponding in conformal field theory to the dressing by the action of a simple current, reflected in the label μ in (2.12).

The basic orientifold action will leave the twisted conjugacy class of the identity (3.3) pointwise invariant, and the orientifold fixed point plane is located in the same place. This is in complete agreement with the discussion of the one-loop amplitude. Similar statements hold for the conjugacy classes (c, c) where c is an element of the center.

The geometry of the other conjugacy classes $\mathcal{C}_{(12)}((g_1, g_2))$ with $(g_1, g_2) \neq (\mathbf{1}, \mathbf{1})$ or (c, c) has been discussed in [31], which we will now review. There is a natural surjection $m : G \times G \to G$ given by group multiplication, $m(g_1, g_2) = g_1g_2$. It follows immediately

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that the twisted conjugacy classes wrapped by the permutation branes get mapped to ordinary conjugacy classes of G. Indeed, one can also easily see the opposite statement, namely that if g_1 and g_2 are conjugate in G then their preimages (g_1', g_1'') with $g_1' g_1'' = g$ and (g_2', g_2'') with $g_2' g_2'' = g_2$ in $G \times G$ are in the same twisted conjugacy class in $G \times G$. The conclusion [31] is that the twisted conjugacy classes are precisely the inverse images under m of the conjugacy classes in G. They are principal G-bundles over the conjugacy classes of G.

In particular, one sees from this point of view that the corresponding boundary states carry the same set of labels as the ordinary Cardy boundary states describing branes wrapping conjugacy classes in G. This is of course in complete agreement with labelling of the permutation boundary states from the conformal field theory point of view (2.8). In addition, we can conclude that the orientifold $g_1 \to g_2^{-1}$ will leave a twisted conjugacy class (g_1, g_2) invariant if $g = g_1 g_2$ is conjugate to its inverse. However, in general the conjugacy class will not be pointwise fixed. As opposed to the case of Cardy branes on a single factor, orientifold actions dressed by an application of an order 2 element of the center will map the twisted conjugacy classes (set-wise) in the same way, though might act differently on the individual elements of the class. We will see examples for this in the case of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ which we will now discuss in detail.

3.1 $\mathrm{SU}(2) \times \mathrm{SU}(2)$

Let us consider the case $\mathrm{SU}(2) \times \mathrm{SU}(2)$. In this case, we will be able to give an interpretation for the orientifold action for all D-branes.

First, we will summarize the results of [16, 15, 14] on the orientifolds of a single $\mathrm{SU}(2)$ model. In this case, there are two orientifolds to consider, parity inversion with a combination of $g \to \pm g^{-1}$. In conformal field theory, this corresponds to the fact that there are two order two simple currents in the model, the identity and the current corresponding to the representation $k/2$. The crosscap states are then given as in (2.13) [2], where $\mu = 0$ or $\mu = k/2$.

We start by discussing the involution $g \to +g^{-1}$, corresponding to the choice $\mu = 0$ on the CFT side. The fixed point set consists of the two group elements $\pm \mathbf{1}$ such that this parity describes a 0-dimensional orientifold. The two orientifold points can have equal or opposite tensions, and, using the properties of the $\mathrm{SU}(2)$ P-matrix, it is easy to see that rational conformal field theory realizes the case of opposite tensions (leading to vanishing total tension) for the case k odd, and equal tension for the case k even. A priori, this tension could be positive or negative, corresponding to an overall sign by which the crosscap can be multiplied.

Turning to the open string sector, D-branes on $\mathrm{SU}(2)$ wrap conjugacy classes. Two conjugacy classes, namely those of $\pm \mathbf{1}$ are pointlike, the others are 2-spheres. The map $g \to g^{-1}$ induces an inversion on each individual two-sphere.

The ground states of the open string spectrum are spherical harmonics, whose behavior under inversion is standard

$$Y_{lm} \to (-1)^l Y_{lm} \ . \tag{3.5}$$

This determines the projection in the open string channel up to an overall sign that can depend on the specific brane and the overall sign of the crosscap.

These geometrical expectations are nicely matched by the explicit evaluation of the conformal field theory Möbius strips, with the result

$$\langle \mathcal{C}, 0 | e^{-\frac{\pi i}{4\tau} H_c} | \mathcal{B}, J \rangle = (-1)^{2J} \sum_{j=0}^{\min\{2J, k-2J\}} (-1)^j \chi_j(\tau) \,, \tag{3.6}$$

where J labels the Cardy boundary state and the sum is taken over integer j. The special values $J = 0$ and $J = k/2$ correspond to the pointlike conjugacy classes, all other values label two-spheres [7, 8].

The other involution $g \to -g^{-1}$ exchanges the two pointlike conjugacy classes and leaves the equatorial 2 sphere fixed. Hence, one expects the involution to act trivially on the open string states on the brane wrapping the equatorial two-sphere. This can be verified from the conformal field theory point of view for the case k even, where the brane with $J = k/4$ wraps the equator. All other branes are mapped to image branes by a reflection at the equator, in terms of conformal field theory, the image of the brane J is $k/2 - J$ as is reflected in the Möbius amplitude

$$\langle \mathcal{C}, \frac{k}{2} | e^{-\frac{\pi i}{4\tau} H_c} | \mathcal{B}, J \rangle = \langle \mathcal{C}, \frac{k}{2} | e^{-\frac{\pi i}{4\tau} H_c} | \mathcal{B}, \frac{k}{2} - J \rangle = \sum_{j=\frac{k}{2}-2J}^{k/2} \chi_j(\tau) \ . \tag{3.7}$$

To generalize this to the case of twisted conjugacy classes of a product of two SU(2)'s, we have to understand the algebra of functions on the twisted conjugacy class, following [9]. It is then possible to interpret the orientifold action derived from the Möbius strip in terms of an involution of that algebra, giving a geometric interpretation of the orientifold action. The authors of [9] consider the open string sector for an arbitrary ω-twisted D-brane on a compact simply connected simple group manifold H, where ω denotes the automorphism. We want to apply their strategy to the case $H = G \times G$, and ω the permutation of the two factors. The twisted boundary conditions of [9] are labelled by representations of the ω- invariant subgroup $H^\omega = \{h \in H | \omega(h) = h\}$. In our case H^ω is the diagonal subgroup G of $G \times G$ and we have seen that indeed the permutation D-branes carry representation labels of G. Quite generally, the open string sector for a pair of twisted conjugacy classes labelled (J_1, J_2) is realized by the H-module $\mathcal{A}^{(J_1,J_2)}$, where

$$\mathcal{A}^{(J_1,J_2)} \sim \mathrm{Inv}_{H^\omega}\left(\mathcal{F}^{(J_1,J_2)}\right) \quad \text{and} \quad \mathcal{F}^{(J_1,J_2)} := \mathcal{F}(H) \otimes \mathrm{Hom}(V_{J_1}, V_{J_2}) \ . \tag{3.8}$$

Here, $\mathcal{F}(H)$ denotes the algebra of functions on the group H and V_{J_i} is a representation space for the irreducible representation J_i. The group $H \times H$ acts on the space of functions $\mathcal{F}(H)$ by the regular action so that we have a natural action of $H \times H^\omega$ on the space of matrix valued functions $\mathcal{F}^{(J_1,J_2)}$ by

$$A^{h_1,h_2}(g) = R_{J_2}(h_2) A(h_1^{-1} g h_2) R_{J_1}(h_2)^{-1}, \tag{3.9}$$

where $h_1 \in H$ $(= G \times G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ in our case) and $h_2 \in H^\omega$ $(= G$ in our case) and R_{J_i} are representation matrices of h_2. The space $\mathcal{A}^{(J_1,J_2)}$ is then the restriction to those matrix valued functions that are invariant under the action of $\mathbf{1} \times H^\omega$ which is $\mathbf{1} \times G$ in our case. To show that this H-module is equivalent to the module of open string ground states determined by the open string partition function, [9] decompose $\mathcal{A}^{(J_1,J_2)}$ into irreducibles. In our case, using the Peter-Weyl theorem

$$\mathcal{F}(G \times G) = \mathcal{F}(G) \otimes \mathcal{F}(G) \equiv \left(\bigoplus_{j_1} V_{j_1} \otimes V_{j_1} \right) \otimes \left(\bigoplus_{j_2} V_{j_2} \otimes V_{j_2} \right), \tag{3.10}$$

where j_i label irreducible representations of SU(2). This space, which is a representation space of $(G \times G)^2$ has to be decomposed with respect to the diagonal G acting in the right regular action, which is easily achieved using the non truncated fusion rules of the Lie group. To obtain $\mathcal{F}^{(J_1,J_2)}$ the result has to be tensored with $V_{J_1} \otimes V_{J_2}$,

$$\mathcal{F}^{(J_1,J_2)} = \bigoplus_{j_1,j_2,j} N_{j_1 j_2}^{j} V_j \otimes V_{j_1} \otimes V_{j_2} \otimes V_{J_1} \otimes V_{J_2} \ . \tag{3.11}$$

To find $\mathcal{A}^{(J_1,J_2)}$ one has to reduce to the G-invariant part, which can again be done using the fusion rules. The result is

$$\mathcal{A}^{(J_1,J_2)} = \bigoplus_{j_1,j_2,j} N_{j_1 j_2}^{j} N_{J_1 J_2}^{j} V_{j_1} \otimes V_{j_2}. \tag{3.12}$$

Comparing to the open string partition function between permutation branes labelled J_1 and J_2

$$\mathcal{C}_{(12)(12)}(J_1, J_2) = \sum_{j,j_1,j_2} N_{J_1 J_2}^{j} N_{j_1 j_2}^{j} \chi_{j_1}(\tau) \chi_{j_2}(\tau) \tag{3.13}$$

we see that these considerations have correctly reproduced the structure of open string ground states. As we have seen [31], the twisted conjugacy classes are principal G-bundles over the conjugacy classes of G. Hence, they locally look like a conjugacy class times the group itself, and for the SU(2) case this is true also globally, so that the geometry of the twisted conjugacy class is $S^2 \times S^3$. One would expect that the parity acts on S^3 either as the identity or the $\mathbb{Z}_2$ anti-podal identification. For the branes wrapping the conjugacy classes $S^3 \times \{pt\}$ one therefore expects a trivial action in the first case, which motivates that the Möbius strip should take the form

$$\mathcal{M}_{(12)(12)}(0,0) = \sum_{j=0}^{\frac{k}{2}} \hat{\chi}_j(\tau) \hat{\chi}_j(\tau) \ . \tag{3.14}$$

This, as mentioned before, is indeed the Möbius strip for the crosscap corresponding to the trivial simple current $\mu = 0$. In the second case one would expect that the open string ground states transform in the same way as when taking a $\mathbb{Z}_2$ orbifold of a single SU(2) to obtain SO(3), that is the expectation for the Möbius strip is

$$\mathcal{M}_{(12)(12)}(\frac{k}{2},0) = \sum_{j=0}^{\frac{k}{2}} (-1)^{2j} \hat{\chi}_j(\tau) \hat{\chi}_j(\tau), \tag{3.15}$$

which is indeed the Möbius strip involving the crosscap with $\mu = \frac{k}{2}$. For the other D-branes, wrapping conjugacy class of the topology $S^3 \times S^2$, this action has to be combined with the action coming from the S^2 part. It can be expected that the latter is simply inherited from the action of the inversion on the conjugacy classes of a single SU(2) factor. As reviewed at the beginning of this section, the latter is an inversion under which the spherical harmonics pick up a sign $(-1)^j$, where j is integer. This leads to the following geometrically motivated expectation for the Möbius strip involving the brane labelled J for the case $\mu = 0$

$$\mathcal{M}_{(12)(12)}(0, J) = \sum_{j,j_1,j_2=0}^{\frac{k}{2}} N_{j_1 j_2}^{j} N_{JJ}^{j} (-1)^{2J+j} \hat{\chi}_{j_1}(\tau)\hat{\chi}_{j_2}(\tau), \tag{3.16}$$

where we have included the same brane-dependent sign $(-1)^{2J}$ that appeared in the Möbius amplitudes for a single SU(2). On the other hand, the Möbius strip derived from conformal field theory is

$$\mathcal{M}_{(12)(12)}(0, J) = \sum_{j_1,j_2=0}^{\frac{k}{2}} Y_{Jj_1}^{j_2} \hat{\chi}_{j_1}(\tau)\hat{\chi}_{j_2}(\tau), \tag{3.17}$$

where we refer to (A.1) in the appendix for the definition of the Y tensor. In order to obtain an agreement between these two expressions, we need the identity, (for $J, j_1, j_2 \leq k/4$)

$$Y_{Jj_1}^{j_2} = \sum_{j=0}^{\frac{k}{2}} N_{JJ}^{j} N_{j_1 j_2}^{j} (-1)^{2J+j}, \tag{3.18}$$

which can indeed be shown to be true using the explicit form for the SU(2) Y-tensor given in the appendix. Likewise, the geometrical expectation for the amplitude between the brane J and the crosscap with label $\mu = \frac{k}{2}$ is

$$\mathcal{M}_{(12)(12)}\left(\frac{k}{2}, J\right) = \sum_{j,j_1,j_2=0}^{\frac{k}{2}} N_{j_1 j_2}^{j} N_{JJ}^{j} (-1)^{2j_1+j+2J} \hat{\chi}_{j_1}(\tau)\hat{\chi}_{j_2}(\tau) \ . \tag{3.19}$$

The prefactor of $\hat{\chi}_{j_1}(\tau)\hat{\chi}_{j_2}(\tau)$ derived from conformal field theory is $Y_{\frac{k}{2}-J,j_1}^{j_2}$, and to get agreement (up to a factor of $(-1)^k$) we use (3.18) and the following symmetry of the Y-tensor of SU(2) (see appendix B)

$$Y_{\frac{k}{2}-J,j_1}^{j_2} = (-1)^{k+2j_1} Y_{Jj_1}^{j_2} \ . \tag{3.20}$$

One might wonder if there also exists a crosscap state exchanging the permutation boundary states $J = 0$ and $J = \frac{k}{2}$, similar to what happened in the case of a single SU(2). This would imply that the conjugacy class of $(\mathbf{1}, \mathbf{1})$ gets mapped to $(\mathbf{1}, -\mathbf{1})$, as would happen under the map $g_1 \to -g_2^{-1}$, $g_2 \to g_1^{-1}$. This parity squares to $-\mathbf{1}$ and in particular is non-involutive. That means that the corresponding crosscap states have to be built on circles twisted by $P^2 \neq \mathbf{1}$. In our case the ground states on which the twisted permutation crosscap state are built differ in the first and second factor by an application of the simple current $\frac{k}{2}$. The crosscap Ishibashi states take the form

$$|\mathcal{C},\Big(j,\frac{k}{2}-j\Big)\rangle\!\rangle_{(12),P^2}=e^{\pi i(L_o^{\rm tot}-h_j^{\rm tot})}\sum_{M,N=0}^{\infty}|j,M\rangle\otimes|\frac{k}{2}-j,N\rangle\otimes U|\frac{k}{2}-j,N\rangle\otimes U|j,M\rangle\ . \tag{3.21}$$

This leads to the following inner product between such permutation crosscap states

$${}_{(12),P^2}\langle\!\langle\mathcal{C},\Big(j,\frac{k}{2}-j\Big)|e^{-\frac{\pi i H_c}{2\tau}}|\mathcal{C},\Big(j',\frac{k}{2}-j'\Big)\rangle\!\rangle_{(12),P^2}=\delta_{j,j'}\chi_j\Big(-\frac{1}{2\tau}\Big)\chi_{\frac{k}{2}-j}\Big(-\frac{1}{2\tau}\Big), \tag{3.22}$$

where the subscript P^2 is supposed to indicate that the circle is twisted by P^2. The full twisted crosscap state

$$|\mathcal{C}\rangle_{(12)P^2}=\sum_{j=0,j\in\mathbb{N}}^{\frac{k}{2}}\Big(|\mathcal{C},(j,j)\rangle\!\rangle_{(12)}+|\mathcal{C},\Big(j,\frac{k}{2}-j\Big)\rangle\!\rangle_{(12)P^2}\Big) \tag{3.23}$$

has a natural interpretation as a crosscap state in the simple current orbifold leading to the D-modular invariant. For $k=0$ mod 4 this is a simple current extension. We have checked that this crosscap state leads to consistent one-loop amplitudes, the results can be found in [32]. For a general description of permutation crosscap states on orbifolds see [28].

To complete the geometrical picture of the permutation orientifold, we show how the localization of the orientifold fixed planes can be determined by scattering localized closed string wave packets, following [20], see also [8, 14, 15, 21]. For the case of permutation D-branes, a similar calculation was done in [21].

The permutation crosscap state associated to the identity simple current is given by the expression involving the Ishibashi crosscap states

$$\begin{aligned}|\mathcal{C},0\rangle_{(12)}&=\sum_{j=0}^{\frac{k}{2}}\frac{S_{0j}}{S_{0j}}|\mathcal{C},j\rangle\!\rangle_{(1\ 2)}\\&=\sum_{j=0}^{\frac{k}{2}}e^{\pi i(L_0-h_j)}\sum_{M_1,M_2=0}^{\infty}\underbrace{|j,M_1\rangle\otimes|j,M_2\rangle}_{\text{left}}\otimes\underbrace{U\,|j,M_2\rangle\otimes U\,|j,M_1\rangle}_{\text{right}}\end{aligned}$$

We now take a closed string state $|g_1,g_2\rangle$ that is localized at $(g_1,g_2)\in\mathrm{SU}(2)\times\mathrm{SU}(2)$. Its expansion coefficients are following [20], appendix D, given by

$$\langle g_1,g_2|\Big(\underbrace{|j_1,m_1,m_2\rangle}_{1^{rmst}\,\mathrm{SU}(2)}\otimes\underbrace{|j_2,n_1,n_2\rangle}_{2^{\rm nd}\,\mathrm{SU}(2)}\Big)\approx\sqrt{2j_1+1}\sqrt{2j_2+1}\mathcal{D}^{j_1}_{m_1,m_2}(g_1)\mathcal{D}^{j_2}_{n_1,n_2}(g_2)$$

where the $\mathcal{D}^j_{m,n}(g)$ are the matrix coefficients of the group element g in the $2j+1$ dimensional representation of SU(2). Here the indices m_1,m_2 label only descendants of the horizontal subalgebra, meaning that we only act with zero modes.

If we now look at the coupling between the crosscap and this localized state and take the approximation of dropping all the descendants appearing in the expression for $|\mathcal{C},0\rangle_{(1\ 2)}$, then we get

$$\begin{aligned}\langle g_1,g_2\,|\mathcal{C},0\rangle_{(1\ 2)} &\approx \sum_{j=0}^{\frac{k}{2}}\sum_{M_1,M_2}\langle g_1,g_2|\left(|j,M_1,M_2\rangle\otimes|j,M_2,M_1\rangle\right)\\ &= \sum_{j=0}^{\frac{k}{2}}\sum_{M_1,M_2}(2j+1)\mathcal{D}^j_{M_1,M_2}(g_1)\mathcal{D}^j_{M_2,M_1}(g_2)\\ &= \sum_{j=0}^{\frac{k}{2}}(2j+1)\mathrm{Tr}(\mathcal{D}^j(g_1)\mathcal{D}^j(g_2)) = \sum_{j=0}^{\frac{k}{2}}(2j+1)\mathrm{Tr}(\mathcal{D}^j(g_3))\end{aligned}$$

where $g_3 = g_1g_2$ by the representation property of the $\mathcal{D}^j$ matrices. If we parametrize SU(2) by the polar angles (ψ,θ,ϕ), then we obtain

$$\langle g_1,g_2\,|\mathcal{C},0\rangle_{(1\ 2)} \approx \sum_{j=0}^{\frac{k}{2}}(2j+1)\frac{\sin((2j+1)\psi_3)}{\sin(\psi_3)} = \sum_{n=1,n\in\mathbb{N}}^{k+1} n\frac{\sin(n\psi_3)}{\sin(\psi_3)} \tag{3.24}$$

which as a function of ψ_3 has a strong peak at zero of height $\sum_{n=1}^{k+1} n^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ and is practically zero elsewhere. Thus we are interested at the points (g_1,g_2) for which the latitude angle of their product is zero. Since these correspond to the north pole, we get the relation $g_1g_2 = 1$, so that the crosscap state is localized on the twined conjugacy class $\mathcal{O}_3^+ = \left\{(g,g^{-1})|g\in \mathrm{SU}(2)\right\}$ as was to be expected from our earlier considerations. This confirms that the permutation crosscap state $|\mathcal{C},0\rangle_{(1\ 2)}$ associated to the identity simple current is localized on the fixed point set of the involution $(g_1,g_2)\mapsto(g_2^{-1},g_1^{-1})$.
Similarly we can look at the permutation crosscap associated to the simple current $\frac{k}{2}$. Since $S_{\frac{k}{2}j} = (-1)^{2j}S_{0j}$, this differs from the previous one only by a factor of $(-1)^{2j}$ in the expansion, meaning that the calculations are practically identical. One obtains

$$\langle g_1,g_2|\mathcal{C},\frac{k}{2}\rangle_{(1\ 2)} \approx \sum_{n=1,n\in\mathbb{N}}^{k+1}(-1)^{n-1}n\frac{\sin(n\psi_3)}{\sin(\psi_3)} \tag{3.25}$$

which no longer has a peak at zero but at $\psi_3 = \pi$. Therefore one now has that $g_1g_2 = -1$, meaning that the crosscap state is localized on the orientifold fixed plane $\mathcal{O}_3^- = \left\{(g,-g^{-1})|g\in \mathrm{SU}(2)\right\}$ of the involution $(g_1,g_2)\mapsto(-g_2^{-1},-g_1^{-1})$.
In the language of twisted conjugacy classes that we developed previously, we have

$$\mathcal{O}_3^+ = \mathcal{C}_{(12)}((\mathbf{1},\mathbf{1})) \quad \text{and} \quad \mathcal{O}_3^- = \mathcal{C}_{(12)}((\mathbf{1},-\mathbf{1}))\ . \tag{3.26}$$

3.2 Rational U(1) × U(1) theories

In this section, we consider D-branes on the torus $T = S^1\times S^1$, where both circles have equal radius $R = \sqrt{k}$, for a positive integer k. As is well known, in this situation one can extend the chiral algebra by exponentials with integer weight, such that the theory becomes rational. The Hilbert space of the theory is for the diagonal modular invariant

$$\mathcal{H} = \bigoplus_{n_1,n_2\in\mathbb{Z}_{2k}} (\mathcal{H}_{n_1} \otimes \mathcal{H}_{n_1}) \otimes (\mathcal{H}_{n_2} \otimes \mathcal{H}_{n_2}) \ . \tag{3.27}$$

There are two kinds of permutation orientifolds to consider, in the first case the $\mathrm{U}(1)$ current J gets mapped to

$$P_B : J^{(1)} \to \bar{J}^{(2)}, \quad J^{(2)} \to \bar{J}^{(1)} \tag{3.28}$$

and the corresponding permutation boundary conditions read

$$J^{(1)}(z) = \bar{J}^{(2)}(\bar{z}), \quad J^{(2)}(z) = \bar{J}^{(1)}(\bar{z}) \tag{3.29}$$

The second case differs by a sign

$$P_A : J^{(1)} \to -\overline{J}^{(2)}, \quad J^{(2)} \to -\overline{J}^{(1)} \tag{3.30}$$

and likewise for the boundary conditions. We will in this paper refer to the two different choices of the parity or boundary conditions as A-type and B-type, in analogy to the $N = 2$ supersymmetric case, where a $\mathrm{U}(1)$ current appears as part of the superconformal algebra.

Since $J^{(i)} = \sqrt{k}\partial X^{(i)}$, where $X^{(i)}$ is the free boson parametrizing the ith circle, one can easily deduce that the branes and orientifold fixed planes are closed curves given by the equation $X^{(1)} = \pm X^{(2)} + const.$ The permutation boundary state takes the form

$$|\mathcal{B}, N\rangle_{(12)} = \sum_n \frac{S_{Nn}}{S_{0n}} |\mathcal{B}, (n, \pm n)\rangle\rangle_{(12)} = \sum_n e^{-\pi i \frac{Nn}{k}} |\mathcal{B}, (n, \pm n)\rangle\rangle_{(12)} \tag{3.31}$$

The sign $\pm$ depends on whether the state is A-type or B-type, apart from that the discussion for the two cases is completely parallel. We already know that geometrically these branes are wrapped on circles intersecting the torus baseline at 45 degrees, and the natural assumption is that the label N determines the intersection point of the D-brane with the torus axes.

If we parametrize the torus by two angles $\phi_1, \phi_2 \in [-\pi, \pi]$, we can apply the same localization procedure as before, just modified appropriately. We obtain

$$\langle e^{i\phi_1}, e^{i\phi_2} | \big(|m\rangle \otimes |-m\rangle\big) \approx e^{im(\phi_1-\phi_2)} \tag{3.32}$$

so that the permutation boundary state with label N is localized on the set

$$\langle e^{i\phi_1}, e^{i\phi_2} | \mathcal{B}, N\rangle^B_{(1\ 2)} \approx \sum_{m=-k+1}^{k} e^{im(\phi_1-\phi_2-\frac{N\pi}{k})} \overset{k\to\infty}{\to} \mathrm{const} \cdot \delta(\phi_1 - \phi_2 - \phi_N) \tag{3.33}$$

where $\phi_N \in [-\pi, \pi]$ and N has been taken to vary appropriately so that in the limit $k \to \infty$, $\frac{N\pi}{k} \to \phi_N$. Hence the permutation D-branes for the B-type boundary conditions are simply closed circles given by the equation $\phi_2 = \phi_1 - \phi_N$. The calculations for the A type are analogous, with the result that $\phi_2 = -\phi_1 + \phi_N$. As we shall see later on, the

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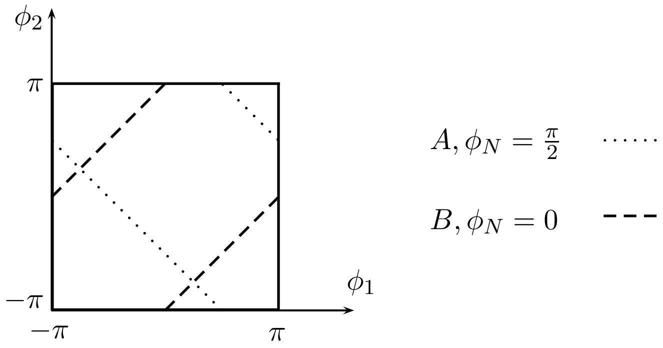

Figure 1: The loci of the A/B D-branes for different values i=of ϕ_N

permutation crosscaps are localized on the same sets as their D-branes counterparts. The one-loop amplitude between two D-branes of equal type is

$$\mathcal{C}_{(12)(12)}(N, M) = \sum_{n=-k+1}^{k} \chi_n(\tau)\chi_{N-M\mp n}(\tau), \tag{3.34}$$

such that open string states for $N \neq M$ have non-trivial windings, which is natural, since those branes are separated by a finite distance. Likewise, we have the crosscap state

$$|\mathcal{C}, \mu\rangle_{(12)} = \sum_n e^{-\pi i \frac{\mu n}{k}} |\mathcal{C}, (n, \pm n)\rangle\!\rangle_{(12)} \tag{3.35}$$

Since the closed string couplings to the ground states are the same as for the D-brane carrying the same label, an immediate conclusion that could be reached by a calculation similar to the previous one is that this crosscap state is located in the same place as the brane. Moreover, the one-dimensional orientifold fixed sets can be written as twisted conjugacy classes:

$$\mathcal{O}_1^A = \mathcal{C}_{(12)}((e^{i\phi_N}, \mathbf{1})) \quad \text{and} \quad \mathcal{O}_1^B = \mathcal{C}_{(12)\circ I}((e^{i\phi_N}, \mathbf{1})), \tag{3.36}$$

where $I : e^{i\phi} \to e^{-i\phi}$ is the inversion. The Möbius amplitude between a crosscap and boundary state of equal type is

$$\mathcal{M}_{(12)(12)}(\mu, M) = \sum_n \hat{\chi}_n(\tau)\hat{\chi}_{2\mu-2M\mp n}(\tau)\sigma(\mu, M, n) \tag{3.37}$$

Here, $\sigma(\mu, M, n)$ is a sign taking the values

$$\sigma(\mu, M, n) = \begin{cases} 1 & \text{if } \mathcal{P}(2\mu - 2M \mp n) = 2\mu - 2M \mp n \bmod 4k \\ (-1)^{n+k} & \text{if } \mathcal{P}(2\mu - 2M \mp n) = 2\mu - 2M \mp n + 2k \bmod 4k. \end{cases} \tag{3.38}$$

where $\mathcal{P}(n) = \hat{n}$ sends the argument to the standard range $\{-k+1, \ldots, k\}$. On general grounds, the multiplicities of the characters appearing in the loop channel between permutation crosscap and boundary state are given by the Y tensor, which we have spelled out here explicitly.

By comparing the Cylinder and Möbius strip we see that the image brane of the brane M is the brane $2\mu - M$. In particular, a brane that remains fixed under the involution has to fulfill $M = 2\mu - M \mod 2k$ with the two solutions $M = \mu$ and $M = \mu + k$. Note that in the two cases in (3.38) are exchanged under $M \to M + k$. In particular, there are no signs for the case $M = \mu$, but a non-trivial orientifold action on the open string states in the case $M = \mu + k$. This indicates that the brane $M = \mu$ is left pointwise fixed, as we discussed earlier. On the other hand, the brane $M = \mu + k$ is left only setwise fixed, the orientifold involution identifies diametrically opposite points of the fixed point circle. The open string states transform accordingly.

4. The $N = 2$ minimal models

4.1 The conformal field theory approach

We consider the tensor product of two $N = 2$ supersymmetric minimal models. A single minimal model with a non-chiral GSO projection has a description in terms of a coset model $su(2)_k \oplus u(1)_2/u(1)_{k+2}$. The representations of the coset algebra are labelled by (j, n, s), where j denotes the spin of the $su(2)$ representation and $n \in \mathbb{Z}_{2k+4}$, $s \in \mathbb{Z}_4$. These labels are subject to a selection rule that requires $2j + n + s$ to be even. Furthermore, one has to identify

$$(j, n, s) \sim \left(\frac{k}{2} - j, n + k + 2, s + 2\right) . \tag{4.1}$$

We will denote the equivalence classes by $[j, n, s]$. We choose the range for the labels of the $\mathrm{U}(1)_{k+2}$ theory to be $-k-1, \ldots, k+2$ and denote the sum of n_1 and n_2 reflected back into that range by $n_1 \hat{+} n_2$. Likewise, our range for the s labels is $-1, 0, 1, 2$, with the same notation for the sum of s-labels.

The conformal weights and $\mathrm{U}(1)$ charges of the highest weight states of the coset representations are, up to integers

$$\begin{aligned} h_{j,n,s} &= h_j - h_n + h_s \mod 1 \qquad (4.2)\\ &= \frac{j(j+1)}{k+2} - \frac{n^2}{4(k+2)} + \frac{s^2}{8} \mod 1 \qquad (4.3)\\ q_{j,n,s} &= \frac{s}{2} - \frac{n}{k+2} \mod 2 \qquad (4.4) \end{aligned}$$

These relations hold exactly in the so-called standard range, where $|n - s| \leq 2j$. In the context of orientifolds, one often needs the square root of the T transformation and therefore the value of h modulo 2. In the coset model, one introduces the sign factors $\sigma_{j,n,s}$ [22]

$$\sigma_{j,n,s} = T^{\frac{1}{2}}_{[j,n,s]} T_j^{-\frac{1}{2}} T_n^{\frac{1}{2}} T_s^{-\frac{1}{2}}, \tag{4.5}$$

where the T matrices with single labels appearing on the right hand side are those of the $su(2)$ and $u(1)$ theories. The explicit value of $\sigma_{j,n,s}$ can be calculated by studying the realization of the coset model ground states within the representation spaces of $su(2)_k \oplus$

$u(1)_2$, see the appendix of [22] for the explicit calculation. The result is

$$\sigma_{j,n,s} = \begin{cases} 1 & \text{for } (j,n,s) \in \text{Standard range} \\ 1 & \text{for } (j,-2j,2), 2j \geq 1 \\ 1 & \text{for } (\frac{k}{2}, k+2, 0) \\ (-1)^{\frac{|n|-|s|}{2}-j} & \text{for } (\frac{k}{2}-j, n\hat{+}(k+2), s\hat{+}2) \in \text{Std. range} \\ -1 & \text{for } (j, 2j+2, 0), j \neq \frac{k}{2} \\ -1 & \text{for } (j, \pm(2j+1), \mp 1) \\ -1 & \text{for } (0,0,2) \end{cases} \tag{4.6}$$

The Hilbert space for the coset model is

$$\mathcal{H} = \bigoplus_{[j,n,s]} \mathcal{H}_{[j,n,s]} \otimes \bar{\mathcal{H}}_{[j,n,\pm s]}, \tag{4.7}$$

where the choice of sign corresponds to two different GSO projections, type 0A and type 0B. On the other hand, the representation spaces of the $N = 2$ super-Virasoro algebra (no GSO projection) are in terms of the representation spaces of the coset theory

$$\mathcal{H}^{N=2}_{[j,n,[s]]} = \mathcal{H}_{[j,n,s]} \oplus \mathcal{H}_{[j,n,s+2]}\,, \tag{4.8}$$

where $[s]$ is the $\mathbb{Z}_2$ reduction of the $\mathbb{Z}_4$ label s and takes the value 1 for the Ramond and 0 for the Neveu-Schwarz sector. In the theory without GSO projection, the fields are organized in these representations of the $N = 2$ algebra.

In this section, we are interested in the crosscap states in the tensor product of two minimal $N = 2$ models with a single GSO projection. The Hilbert space of such a theory is

$$\begin{aligned} \mathcal{H} = \bigoplus_{[j_1,n_1,s_1],[j_2,n_2,s_2]} & \Big((\mathcal{H}_{[j_1,n_1,s_1]} \otimes \mathcal{H}_{[j_2,n_2,s_2]}) \otimes (\bar{\mathcal{H}}_{[j_1,n_1,s_1]} \otimes \bar{\mathcal{H}}_{[j_2,n_2,s_2]}) \\ & \oplus\ (\mathcal{H}_{[j_1,n_1,s_1]} \otimes \mathcal{H}_{[j_2,n_2,s_2]}) \otimes (\bar{\mathcal{H}}_{[j_1,n_1,s_1+2]} \otimes \bar{\mathcal{H}}_{[j_2,n_2,s_2+2]}) \Big)\,, \end{aligned} \tag{4.9}$$

where the sums over s_1 and s_2 are restricted to $s_1 - s_2 \in 2\mathbb{Z}$. In comparison to the tensor product of two coset models, this theory is spin aligned, so that the states of both factors are either in the NSNS or RR sector. This projection gives rise to a twisted sector, where the s_i values for the left- and right movers differ by 2, corresponding to the lower line of (4.9).

The formulas we have developed so far for permutation crosscap states are directly applicable to a tensor product of two coset models, where the Hilbert space is given by the tensor product of two copies of (4.7). The simple currents of a single copy of the coset theory carry representation labels $(0, \nu, \sigma)$, and hence we can label the crosscap states by (ν, σ). For the tensor product of two coset models, the crosscap state would thus read

$$|\mathcal{C}, \nu, \sigma\rangle^{\text{coset}}_{(12)} = \sum_{[j,n,s]} \frac{S_{[0,\nu,\sigma][j,n,s]}}{S_{[0,0,0][j,n,s]}} |\mathcal{C}, [j,n,s][j,\pm n,\pm s]\rangle\!\rangle_{(12)}\ . \tag{4.10}$$

The choice of sign in the Ishibashi state is just like in the U(1) case and refers to different boundary conditions, A-type (+), and B-type (−). We will specialize to the B-type case, the discussion for A-type permutation crosscap states is completely analogous.

Note that the crosscap state for the tensor product of two coset models already obeys the spin alignment condition $s_1 - s_2 \in 2\mathbb{Z}$. To understand how it has to be modified for the GSO projected tensor product of two minimal $N = 2$ models, we consider the Klein bottle amplitude calculated from (4.10)

$$\mathcal{K}^{\mathrm{coset}}_{(12)}((\nu,\sigma),(\nu',\sigma')) = \sum_{[j_i,n_i,s_i]} \delta_{j_1 j_2}\ \delta^{(2k+4)}_{n_1+\nu'-\nu,n_2}\delta^{(4)}_{s_1+\sigma'-\sigma,s_2}\ \chi_{[j_1,n_1,s_1]}(2\tau)\ \chi_{[j_2,n_2,s_2]}(2\tau)\ . \tag{4.11}$$

This Klein bottle amplitude obviously only gets contributions from the first part of the Hilbert space (4.9). The orientifold action in the closed string sector of this model can be read off to be (for $\nu' = \nu, \sigma' = \sigma$) a combination of a permutation and a parity action in a single coset model. On the closed string ground states it acts as

$$\begin{aligned}(j_1,n_1,s_1)\otimes(j_2,n_2,s_2)\otimes\overline{(j_1,n_1,s_1)}\otimes\overline{(j_2,n_2,s_2)} \rightarrow\\ (j_2,n_2,s_2)\otimes(j_1,n_1,s_1)\otimes\overline{(j_2,n_2,s_2)}\otimes\overline{(j_1,n_1,s_1)}\ .\end{aligned} \tag{4.12}$$

The parity invariant states are those with $j_1 = j_2$, $n_1 = n_2$, $s_1 = s_2$, leading to the Klein bottle (4.11). The natural extension of this parity action to the twisted sector part is given by

$$\begin{aligned}(j_1,n_1,s_1)\otimes(j_2,n_2,s_2)\otimes\overline{(j_1,n_1,s_1+2)}\otimes\overline{(j_2,n_2,s_2+2)} \rightarrow\\ (j_2,n_2,s_2+2)\otimes(j_1,n_1,s_1+2)\otimes\overline{(j_2,n_2,s_2)}\otimes\overline{(j_1,n_1,s_1)}\,,\end{aligned} \tag{4.13}$$

such that state with $j_1 = j_2$, $n_1 = n_2$, $s_1 = s_2 + 2$ are parity invariant.

The crosscap states that implement this type of parity action on the closed string sector states is given by projections of the coset model crosscap states onto the NSNS (or RR) sector

$$\begin{aligned}|\mathcal{C},\nu,\sigma\rangle^{NSNS}_{(12)} &= \sum_{[j,n,s]\,s\,\mathrm{ev}} \frac{S_{[0,\nu,\sigma][j,n,s]}}{S_{[0,0,0][j,n,s]}}|\mathcal{C},[j,n,s][j,-n,-s]\rangle\!\rangle_{(12)}\\ |\mathcal{C},\nu,\sigma\rangle^{RR}_{(12)} &= \sum_{[j,n,s]\,s\,\mathrm{odd}} \frac{S_{[0,\nu,\sigma][j,n,s]}}{S_{[0,0,0][j,n,s]}}|\mathcal{C},[j,n,s][j,-n,-s]\rangle\!\rangle_{(12)}\ .\end{aligned} \tag{4.14}$$

The corresponding Klein bottles are

$$\begin{aligned}\mathcal{K}^{NSNS}_{(12)}((\nu,\sigma),(\nu',\sigma')) = \sum_{[j_i,n_i,s_i]} \delta_{j_1 j_2}\ \delta^{(2k+4)}_{n_1+\nu'-\nu,n_2} \times\\ \left(\delta^{(4)}_{s_1+\sigma'-\sigma,s_2} + \delta^{(4)}_{s_1+\sigma'-\sigma+2,s_2}\right)\ \chi_{[j_1,n_1,s_1]}(2\tau)\ \chi_{[j_2,n_2,s_2]}(2\tau)\end{aligned} \tag{4.15}$$

$$\begin{aligned}\mathcal{K}^{RR}_{(12)}((\nu,\sigma),(\nu',\sigma')) = \sum_{[j_i,n_i,s_i]} \delta_{j_1 j_2}\ \delta^{(2k+4)}_{n_1+\nu'-\nu,n_2} \times\\ \left(\delta^{(4)}_{s_1+\sigma'-\sigma,s_2} - \delta^{(4)}_{s_1+\sigma'-\sigma+2,s_2}\right)\ \chi_{[j_1,n_1,s_1]}(2\tau)\ \chi_{[j_2,n_2,s_2]}(2\tau)\end{aligned} \tag{4.16}$$

For $\sigma = \sigma'$, $\nu = \nu'$ the Klein bottle in the NSNS sector encodes the parity action (4.12) and (4.13) in the closed string loop channel. The RR sector crosscap state leads in this case to an additional insertion of $(-1)^{\frac{s_1-2_2}{2}}$, which is the difference in fermion number in the two minimal model factors. [2]

There is another obvious modification of the parity action, namely we can dress it by space-time fermion number, which is -1 on the RR sector states, and $+1$ on the NSNS sector states.

$$\begin{aligned}(j_1,n_1,s_1)\otimes(j_2,n_2,s_2)\otimes\overline{(j_1,n_1,s_1)}\otimes\overline{(j_2,n_2,s_2)} \to \\ (-1)^{s_2}\ (j_2,n_2,s_2)\otimes(j_1,n_1,s_1)\otimes\overline{(j_2,n_2,s_2)}\otimes\overline{(j_1,n_1,s_1)}\end{aligned} \tag{4.17}$$

and likewise its extension to the twisted sector. The crosscap states leading to this modification contain only crosscap Ishibashi states that are built on ground states coming from the twisted sector where $s_i = \bar{s}_i + 2$ for $i = 1, 2$. The corresponding crosscap states are

$$\begin{aligned}|\mathcal{C},\nu,\sigma\rangle^{NSNS}_{(12)} &= \sum_{\substack{[j,n,s]\\ s\,\text{even}}} \sqrt{\frac{T_{j,-n,-(s\hat{+}2)}}{T_{j,-n,-s}}}\,\frac{S_{[0,\nu,\sigma][j,n,s]}}{S_{[0,0,0][j,n,s]}}|\mathcal{C},[j,n,s][j,-n,-s]\rangle\rangle_{(12)} \\ |\mathcal{C},\nu,\sigma\rangle^{RR}_{(12)} &= \sum_{\substack{[j,n,s]\\ s\,\text{odd}}} \sqrt{\frac{T_{j,-n,-(s\hat{+}2)}}{T_{j,-n,-s}}}\,\frac{S_{[0,\nu,\sigma][j,n,s]}}{S_{[0,0,0][j,n,s]}}|\mathcal{C},[j,n,s][j,-n,-s]\rangle\rangle_{(12)}\end{aligned} \tag{4.18}$$

The (jns)-dependent phase factors introduced in front of the Ishibashi states drop out from the Klein bottle calculation but will be motivated in a moment.

Let us now turn to a discussion of the open string one loop amplitudes. The B-type permutation branes are characterized by the gluing conditions

$$\begin{aligned}\left(L^{(1)}_n - \bar{L}^{(2)}_{-n}\right)|\mathcal{B}\rangle\rangle &= \left(L^{(2)}_n - \bar{L}^{(1)}_{-n}\right)|\mathcal{B}\rangle\rangle = 0 \\ \left(J^{(1)}_n + \bar{J}^{(2)}_{-n}\right)|\mathcal{B}\rangle\rangle &= \left(J^{(2)}_n + \bar{J}^{(1)}_{-n}\right)|\mathcal{B}\rangle\rangle = 0 \\ \left(G^{\pm(1)}_r + i\eta_1\bar{G}^{\pm(2)}_{-r}\right)|\mathcal{B}\rangle\rangle &= \left(G^{\pm(2)}_r + i\eta_2\bar{G}^{\pm(1)}_{-r}\right)|\mathcal{B}\rangle\rangle = 0\,.\end{aligned} \tag{4.19}$$

For A-type permutation branes, one would introduce a minus sign in the condition for the U(1) current J, and hence the conditions on the supercharges would have to be modified accordingly.

The B-type permutation boundary states [33, 34] for the GSO projected theory are given by

$$\begin{aligned}|\mathcal{B},J,N,S_1,S_2\rangle_{(12)} &= \frac{1}{\sqrt{2}}\sum_{[j,n,s]}\frac{S_{[J,N,S_2-S_1][j,n,s]}}{S_{[0,0,0],[j,n,s]}}\ \times \\ &\times\left(|\mathcal{B},([j,n,s][j,-n,-s])\rangle\rangle_{(12)} + (-1)^{S_2}|\mathcal{B},([j,n,s][j,-n,-(s\hat{+}2)])\rangle\rangle_{(12)}\right)\,.\end{aligned} \tag{4.20}$$

[2] Note that this type of argument does not fix the phases appearing in front of the crosscap Ishibashi states, since any such phase drops out from our calculation.

The labels $[J,N,S_1,S_2]$ are subject to the constraint $2J+N+S_1-S_2$ even. S_1 and S_2 are $\mathbb{Z}_4$ labels, and S_1-S_2 has to be even to preserve the diagonal $N=2$. Compared to equation (4.19), $(-1)^{S_1}=\eta_1$ and $(-1)^{S_2}=\eta_2$. The boundary state is subject to the identification $[J,N,S_1,S_2]=[J,N,S_1+2,S_2+2]$, and shifting either S_1 or S_2 by 2 corresponds to mapping brane to anti-brane. The open string partition function has been given in [33] and reads

$$
\begin{aligned}
&\mathcal{C}_{(12)(12)}([J,N,S_1,S_2],[\hat{J},\hat{N},\hat{S}_1,\hat{S}_2])(\tau)=\sum_{[j_i,n_i,s_i]}\chi_{[j_1,n_1,s_1]}(\tau)\,\chi_{[j_2,n_2,s_2]}(\tau)\times\\
&\quad\sum_{\hat{j}}\Big[N_{\hat{j}\hat{j}}{}^{J}\,N_{j_1j_2}{}^{\hat{j}}\,\delta^{(2k+4)}(\Delta N+n_1-n_2)\\
&\qquad\times\Big(\delta^{(4)}(\Delta S_1+s_1)\,\delta^{(4)}(\Delta S_2+s_2)+\delta^{(4)}(\Delta S_1+2+s_1)\,\delta^{(4)}(\Delta S_2+2+s_2)\Big)\\
&\qquad+N_{\hat{j}\,\frac{k}{2}-\hat{j}}{}^{J}\,N_{j_1j_2}{}^{\hat{j}}\,\delta^{(2k+4)}(\Delta N+k+2+n_1-n_2)\\
&\qquad\times\Big(\delta^{(4)}(\Delta S_1+2+s_1)\,\delta^{(4)}(\Delta S_2+s_2)+\delta^{(4)}(\Delta S_1+s_1)\,\delta^{(4)}(\Delta S_2+2+s_2)\Big)\Big]\,,
\end{aligned}
$$

where $\Delta N=\hat{N}-N$ and $\Delta S_i=\hat{S}_i-S_i$.

Using the properties of the P and Y matrix of the minimal model listed in the appendix, we now derive the following Möbius strip for the NSNS crosscap in the untwisted sector

$$
\begin{aligned}
&\mathcal{M}^{NSNS}_{(12)(12)}((\nu,\sigma)(J,N,S_1,S_2))(\tau)= \qquad\qquad (4.21)\\
&\quad\sum_{[j_i,n_i,s_i]}\sigma_{j_1,n_1,s_1}\sigma_{j_2,n_2,s_2}\delta^{(2)}_{s_1}\ \delta^{(2)}_{s_2}\hat{\chi}_{[j_1,n_1,s_1]}(\tau)\hat{\chi}_{[j_2,n_2,s_2]}(\tau)\times\\
&\quad\left(Y^{j_2}_{Jj_1}\ Y^{n_2}_{N-\nu\ n_1}\ \delta^{(4)}_{2(S_2-S_1-\sigma)+s_1-s_2}+Y^{\frac{k}{2}-j_2}_{Jj_1}Y^{n_2\hat{+}(k+2)}_{N-\nu\ n_1}\delta^{(4)}_{2(S_2-S_1-\sigma)+s_1-s_2+2}(-1)^{\frac{|n|-|s|-2j}{2}}\right)
\end{aligned}
$$

One sees that for the $\mathrm{SU}(2)$ and $\mathrm{U}(1)_{k+2}$ part of the theory the orientifold action is inherited from that of the constituent theories. In particular, the brane labels get mapped as

$$J\to J,\qquad N\to 2\nu-N\ . \tag{4.22}$$

It remains to analyze the labels of the $\mathrm{U}(1)_2$ part. By comparison of the Möbius strip with the cylinder amplitude one concludes that

$$\sigma=0: S_i\to S_i\ . \tag{4.23}$$

The S_i labels of the boundary state remain unaffected. On the other hand, for the crosscaps with $\sigma=1$ one obtains

$$\sigma=1: S_1\to S_1+2,\quad S_2\to S_2, \tag{4.24}$$

where equivalently one can exchange the roles of S_1 and S_2 by the freedom to shift both boundary state labels by 2. To summarize, the crosscap with $\sigma=0$ maps branes to branes and anti-branes to antibranes, the crosscap with $\sigma=1$ exchanges branes and anti-branes. In a geometrical context, one would interpret such a brane-antibrane flip as an orientation reversal of the cycle wrapped by the brane.

For completeness, we also list the Möbius amplitude for the RR crosscap. The result is

$$\mathcal{M}^{RR}_{(12)(12)}((\nu,\sigma)(J,N,S_1,S_2)) = \sum_{j_i,n_i,s_i} \sigma_{j_1 n_1 s_1}\sigma_{j_2 n_2 s_2} \times \tag{4.25}$$

$$\left(\delta^{(4)}_{S_2-S_1-\sigma+\frac{s_1-s_2}{2}} - \delta^{(4)}_{S_2-S_1-\sigma+\frac{s_1-s_2}{2}+2}\right) \delta^{(2)}_{s_1+1}\, \delta^{(2)}_{s_2+1} Y^{j_2}_{Jj_1} Y^{n_2}_{N-\nu n_1} \hat{\chi}_{(j_1,n_1,s_1)}(\tau)\hat{\chi}_{(j_2,n_2,s_2)}(\tau) \ .$$

In particular, the open string sector is in the R sector, which means that the parity action exchanges even S_i labels of the boundary state with odd ones. A closer look at the amplitudes reveals that

$$\sigma = 0: \quad S_1 \to S_1 + 1, \quad S_2 \to S_2 + 1 \tag{4.26}$$

$$\sigma = 1: \quad S_1 \to S_1 + 1 \quad S_2 \to S_2 - 1, \tag{4.27}$$

so that we see the brane-antibrane flip also here.

Let us now turn to the amplitudes in the twisted sector. Before going into the details, we can already predict what the difference between the twisted and untwisted sector case is: In the closed string sector, the two parity actions differ by the operation that acts as -1 on the RR states and $+1$ on the NSNS. On the level of boundary states, this means that the NSNS part of the boundary state is unaffected, whereas the RR part is transformed with a $-$ sign. This means that the action of this parity on the branes differs from the one discussed previously by a brane-antibrane flip. So one would conclude that in the twisted NSNS sector

$$\sigma = 0: S_1 \to S_1 + 2, \quad S_2 \to S_2\,, \tag{4.28}$$

whereas

$$\sigma = 1: S_i \to S_i + 2\ . \tag{4.29}$$

Similar arguments can be made for the RR sector.

Recall that ν has to be even in the case $\sigma = 0$ and odd in the case $\sigma = 1$, such that there is a difference in the possible parity actions between the untwisted and twisted case.

Let us outline briefly how to obtain these results technically. First, we need to motivate the extra phases appearing in the crosscap state in the twisted sector. For this note that in the fully supersymmetric theory (where the Hilbert space is given by (4.8)) Ishibashi states from the twisted and untwisted sector have to be combined to supersymmetric Ishibashi states. For the boundary state, the supersymmetric Ishibashi states can be directly read off from the boundary state (4.20), explicitly

$$\begin{aligned} |\mathcal{B},([j,n,s][j,-n,-s])\rangle\rangle^{N=2}_{(12)} = \Big(&|\mathcal{B},[j,n,s][j,-n,-s]\rangle\rangle_{(12)} \\ &+(-1)^{S_2}|\mathcal{B},[j,n,s][j,-n,-(s\hat{+}2)]\rangle\rangle_{(12)} \\ &+(-1)^{S_1}|\mathcal{B},[j,n,s\hat{+}2][j,-n,-s]\rangle\rangle_{(12)} \\ &+(-1)^{S_1+S_2}|\mathcal{B},[j,n,s\hat{+}2][j,-n,-(s\hat{+}2)]\rangle\rangle_{(12)}\Big), \end{aligned} \tag{4.30}$$

The conditions to be fulfilled by the crosscap states can be obtained from those for the boundary states (4.19) by conjugation with $e^{\pi i L_0}$, where $L_0 = L_0^{(1)} + L_0^{(2)}$. This conjugate condition is fulfilled by

$$e^{\pi i L_0}|\mathcal{B}\rangle\rangle = e^{\pi i h}|\mathcal{C}\rangle\rangle \tag{4.31}$$

Applying $e^{\pi i L_0}$ to the supersymmetric boundary Ishibashi state shows that for the $N = 2$ supersymmetric crosscap Ishibashi state there has to be a relative phase equal to $\sqrt{T_{[j,-n,-(s\hat{+}2)]}T^{-1}_{[j,-n,-s]}}$ between the crosscap Ishibashi states $|\mathcal{C},[j,n,s][j,-n,-s]\rangle\rangle$ and $|\mathcal{C},[j,n,s][j,-n,-(s\hat{+}2)]\rangle\rangle$, as we have anticipated in (4.18). In the calculation of the Möbius strip, one makes use of the following symmetry property of the P-matrix of the $N = 2$ minimal model

$$P_{[j,n,s][j',n',s'+2]}\frac{\sqrt{T_{[j',n',s']}}}{\sqrt{T_{[j',n',s'+2]}}} = \frac{\sqrt{T_{[j,n,s]}}}{\sqrt{T_{[j,n,s+2]}}}P_{[j,n,s+2][j',n',s']}\ . \tag{4.32}$$

Effectively, this shifts the $\delta^{(4)}$ functions for the s labels in the Möbius strip (4.21) by 2 verifying the action (4.28), (4.29) .
We refer to [28] for further discussions of the $N = 2$ minimal models and their orbifolds. In addition, applications to Gepner models and the construction of string vacua can be found in that paper.

4.2 The Landau Ginzburg approach

A Landau Ginzburg model of chiral superfields X_i has the following action

$$S = \int d^4\theta K(X_i,\bar{X}_i) + \int d\theta^- d\theta^+ W(X_i)|_{\bar{\theta}^\pm=0} + \int d\bar{\theta}^+ d\bar{\theta}^- \bar{W}(\bar{X}_i)|_{\theta^\pm=0} \tag{4.33}$$

Here, $\pm$ distinguishes left and right movers, whereas bar is the complex conjugation. Parity symmetries of Landau Ginzburg theories have been discussed in [22, 24]. In particular, B-type parity acts on the superspace coordinates as $\theta^\pm \to \theta^\mp$, $\bar{\theta}^\pm \to \bar{\theta}^\mp$. Obviously, the measure in the F-term of the action picks up a sign under B-parity. For the action to be invariant, one must define an action on the superfields, such that W flips sign under the induced action

For the case of a single minimal model, the superpotential is

$$W = X^{k+2} \tag{4.34}$$

and the action $X \mapsto -X$ leads to a parity invariant F-term in the case that k is odd. For the case k even, there is no pure involutive B-type parity.

For the tensor product theory, the relevant superpotential is

$$W = X_1^{k+2} + X_2^{k+2} \tag{4.35}$$

and a suitable involution is

$$\tau : X_1 \mapsto \eta X_2, \qquad X_2 \mapsto \eta^{-1} X_1, \tag{4.36}$$

where η is a $(k+2)$th root of -1, $\eta^{k+2} = -1$. So for the tensor product theory, there exists a B-parity for any k, which is what we would have expected from our conformal field theory analysis, where we have explicitly constructed B-crosscap states for any level. Furthermore, it is suggestive that the choice of η corresponds to the choice of the crosscap state label ν in conformal field theory.

That this is indeed the case can be understood best in the language of matrix factorizations [24]. Recall that B-type D-branes in Landau Ginzburg models are described by matrix factorizations of the superpotential [27, 35–38]

$$W = E(X_i)F(X_i), \tag{4.37}$$

where E, F are matrices with polynomial entries. The two polynomial matrices are arranged into a larger matrix that plays the role of a BRST operator for the theory with boundary

$$Q = \begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix} . \tag{4.38}$$

Two factorizations are equivalent, if they are related by a similarity transformation

$$\hat{Q} = UQU^{-1} , \tag{4.39}$$

where both U and U^{-1} are block diagonal matrices with polynomial entries. In our case, the superpotential can be written as

$$W = X_1^{k+2} + X_2^{k+2} = \prod_\eta (X_1 - \eta X_2), \tag{4.40}$$

where η runs over all $(k+2)$th roots of -1. Suitable factorization are then obtained by organizing the linear factors into two groups. These factorizations and their geometric relevance have been studied in [39, 40]. In [33, 34, 41] a subclass of this type was identified with permutation D-branes, namely

$$|\mathcal{B}, J, N, S_1 = S_2 = 0\rangle \Leftrightarrow F = \prod_{m=(N-2J)/2)}^{(N+2J)/2} (X_1 - \eta_m X_2), \tag{4.41}$$

where $\eta_m = e^{-\frac{\pi i(2m+1)}{k+2}}$, and the spin structures were taken to be fixed, such that $2J + N$ even. It remains to be clarified how to obtain conformal field theory descriptions for the other factorizations obtained by grouping the factors in (4.40) in arbitrary ways.

In [24] it was realized that also orientifolds have a description in terms of matrix factorizations. Namely, they are described by the matrix factorization that corresponds to the D-brane localized on the fixed point set. The (topological) crosscap state is the same as the boundary state. In our case, the fixed point set of the involution (4.36) is given by

$$X_1 - \eta X_2 = 0 , \tag{4.42}$$

to which one associates the matrix factorization with

$$F = X_1 - \eta X_2 \tag{4.43}$$

According to the identification of boundary states with matrix factorizations, this brane has boundary state label $J = 0$ and N (which has to be even) is determined by the phase $\eta(N) = \exp(\pi i(N+1)/(k+2))$. It is therefore suggestive that one of the CFT crosscap states labelled ν with ν even corresponds on the LG side to the involution (4.36)

$$\eta = \eta(\nu) = e^{-\frac{\pi i(\nu+1)}{k+2}} \ . \tag{4.44}$$

To see this more precisely, we will consider the action of the parity on the D-branes in Landau Ginzburg language. In [24] (drawing from the unpublished work [42]) it was shown that the parity induces the following operation on the boundary BRST operators

$$Q(X_i) \to -Q(\tau(X_i))^T, \tag{4.45}$$

where $(-)^T$ is the graded transpose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix} , \tag{4.46}$$

where $(-)^t$ denotes the ordinary transpose. The action is indeed quite natural [24]: If one associates Chan Paton spaces V and W to the left and right boundary of an open string, the space of open string states includes the Chan Paton factor

$$\mathcal{H}^{\mathrm{CP}} = \mathrm{Hom}(V, W) \tag{4.47}$$

A parity action exchanges the left and right boundary of the open string. If left and right boundary were initially oppositely oriented, then the new (after application of the parity) left boundary is oppositely oriented compared to the initial left boundary. The space of Chan-Paton factors is mapped to $\mathrm{Hom}(W^*, V^*)$ which is naturally implemented on the open string states by taking the transpose. In the case that the vector spaces are graded, as is the case for Landau Ginzburg models, it is suggestive that a graded version of the transpose should be used. This was indeed confirmed in [24] by analyzing the parity action with the help of the action for the boundary fermions appearing in the Landau Ginzburg action.

This parity acts on the factors $E(X_i)$ and $F(X_i)$ as $F_{\mathrm{image}} = -\tau^* E$, and $E_{\mathrm{image}} = \tau^* F$, such that $F_{\mathrm{image}} E_{\mathrm{image}} = -W$ is a factorization of the parity-transformed superpotential. For the factorization with F as in (4.41) this means that the image factorization under the parity (4.36) is given as

$$E_{\mathrm{image}} = -\prod_{n=(N-2J)/2)}^{(N+2J)/2} (\eta X_2 - \eta_n \eta(\nu)^{-1} X_1) = -\prod_{n=(N-2J)/2)}^{(N+2J)/2} (-\eta^{-1}\eta_n)\ (X_1 - \eta^2 \eta_n^{-1} X_2) \ . \tag{4.48}$$

Parametrizing the possible η by even integers ν as in (4.44) allows us to write the image brane as

$$E_{\mathrm{image}} = \mathrm{const} \times \prod_{n=(2\nu-N+2J)/2}^{(2\nu-N-2J)/2} (X_1 - \eta_n X_2) \ . \tag{4.49}$$

This is, up to an exchange of the factors E and F which corresponds to an orientation flip, indeed the factorization associated to the boundary state with $N \to 2\nu - N$, $J \to J$. The action $N \to 2\nu - N$ and $J \to J$ is indeed the parity action that we also observed in conformal field theory. Note however that in conformal field theory, depending on the value of the crosscap label σ, ν could take even or odd labels. In the Landau Ginzburg theory, the ν labels are even (with the choices we have made), in particular, the reflection $N \to -N$ is a possible parity action on the brane labels. The natural action on the Chan-Paton factors by a graded transpose induces an orientation flip of the Landau-Ginzburg brane. In the conformal field theory, there was a choice of parities, one of them leading to a brane-anti-brane flip, the other not. Here we have seen that the parity with flip corresponds directly to the Landau Ginzburg parity. To conclude, the parity action on brane labels in (4.28) together with $J \to J$ and $N \to 2\nu - N$, ν even is the CFT description (in the NSNS sector) of the Landau Ginzburg parity.

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A. Loop amplitudes

First, we define the following set of matrices from the fusion and Y coefficients

$$(N_i)_{mn} := N_{im}^n = \sum_j \frac{S_{ij} S_{mj} \bar{S}_{nj}}{S_{0j}} \quad \text{and} \quad (Y_i)_{mn} := Y_{im}^n = \sum_j \frac{S_{ij} P_{mj} \bar{P}_{nj}}{S_{0j}} \tag{A.1}$$

where S are the usual modular matrices and $P := \sqrt{T} S T^2 S \sqrt{T}$. Tensor boundary and crosscap states are denoted with an additional subscript 1 and carry a boundary index α composed of two labels $\alpha = (\alpha_1, \alpha_2)$, whereas the permutation states are denoted with the transposition (1 2) and with only one boundary label α. In the following, α and β are boundary state labels, whereas μ and ν refer to crosscaps, and are in particular simple currents. Then, the one loop amplitudes for the conjugate modular invariant are expressed, first for the cylinder case, as

$$\begin{aligned}
\mathcal{C}_{11}(\alpha,\beta) &= \sum_i \chi_i(\tau)(N_i)_{\beta_1\alpha_1} \sum_j \chi_j(\tau)(N_j)_{\beta_2\alpha_2} \\
\mathcal{C}_{1(1\ 2)}(\alpha,\beta) &= \sum_i \chi_i\left(\frac{\tau}{2}\right)(N_i N_{\bar{\alpha}_1})_{\beta\alpha_2} \\
\mathcal{C}_{(1\ 2)(1\ 2)}(\alpha,\beta) &= \sum_{i,j} \chi_i(\tau)\chi_j(\tau)(N_i N_j)_{\beta\alpha}
\end{aligned} \tag{A.2}$$

As for the Klein bottle amplitudes, the results are

$$\begin{aligned}
\mathcal{K}_{11}(\mu,\nu) &= \sum_i \chi_i(2\tau)(Y_i)_{\nu_1\mu_1} \sum_j \chi_j(2\tau)(Y_j)_{\nu_2\mu_2} \\
\mathcal{K}_{1(1\ 2)}(\mu,\nu) &= \sum_i \chi_i(\tau)(Y_i Y_\nu)_{\bar{\mu}_1\mu_2} \\
\mathcal{K}_{(1\ 2)(1\ 2)}(\mu,\nu) &= \sum_{i,j} \chi_i(2\tau)\chi_j(2\tau)(N_j N_i)_{\nu\mu}
\end{aligned} \tag{A.3}$$

Finally, we obtain for the Möbius strip case the results

$$\begin{aligned}
\mathcal{M}_{11}(\mu,\alpha) &= \sum_i \hat{\chi}_i(\tau)(Y_{\alpha_1})_{i\mu_1} \sum_j \hat{\chi}_j(\tau)(Y_{\alpha_2})_{j\mu_2} \\
\mathcal{M}_{(1\ 2)1}(\mu,\alpha) &= \sum_i \hat{\chi}_i(2\tau)(N_{\alpha_1}N_{\alpha_2})_{i\mu} \\
\mathcal{M}_{1(1\ 2)}(\mu,\alpha) &= \sum_i \hat{\chi}_i(2\tau)(Y_\alpha Y_i)_{\bar{\mu}_1\mu_2} \\
\mathcal{M}_{(1\ 2)(1\ 2)}(\mu,\alpha) &= \sum_{i,j} \hat{\chi}_i(\tau)\hat{\chi}_j(\tau)(Y_\alpha Y_{\bar{\mu}})_{i\bar{j}},
\end{aligned} \tag{A.4}$$

where we remark that in our convention $\mathcal{M} = \langle \mathcal{C}| e^{-\frac{\pi i H_c}{4\tau}} |\mathcal{B}\rangle$.

B. The Y tensor of SU(2)

The coefficients of the Y tensor in the case of $\widehat{su}_k(2)$ are given by

$$\begin{aligned}
Y_{ij}^n = \delta_{2i}^{(2)}\delta_{jn} + \sum_{l=1,2i+l\in 2\mathbb{N}}^{2i} \Big[&\delta_{j+l,n} + \delta_{j,n+l} - \delta_{j+n+1,l} \\
&-(-1)^{2j+k}(\delta_{j+l,n+k+2} + \delta_{j+k+2,n+l} - \delta_{j+n+l,k+1})\Big]
\end{aligned} \tag{B.1}$$

In particular, these formulas imply that Y_{ij}^n is zero whenever $j \in \mathbb{N}$ and $n \in \mathbb{N} + \frac{1}{2}$ or $j \in \mathbb{N} + \frac{1}{2}$ and $n \in \mathbb{N}$. This means that $(-1)^{2j} = (-1)^{2n}$ so the Y_i matrices are symmetric in j, n as they should be.

Of great utility to us are going to be the following simpler expressions

$$\begin{aligned}
&Y_{i0}^0 = (-1)^{2i} \qquad Y_{i\frac{k}{2}}^{\frac{k}{2}} = 1\ \forall i \qquad Y_{i\frac{k}{2}}^0 = \frac{1+(-1)^k}{2}\delta_{i\frac{k}{4}} = \delta_k^{(2)}\delta_{i\frac{k}{4}} \\
&Y_{\frac{k}{4}i}^{\frac{k}{2}} = \delta_k^{(2)}\delta_{2i}^{(2)} \qquad Y_{\frac{k}{4}i}^0 = \delta_k^{(2)}\delta_{2i}^{(2)}(-1)^{i+\frac{k}{2}}
\end{aligned} \tag{B.2}$$

Similarly, the Y tensor has the symmetry relations

$$\begin{aligned}
Y_{ij}^n &= Y_{in}^j & Y_{i\frac{k}{2}-j}^{\frac{k}{2}-n} &= (-1)^{2i+j-n}Y_{ij}^n \\
Y_{\frac{k}{2}-i\ j}^n &= (-1)^{k+2j}Y_{ij}^n & Y_{i\frac{k}{2}-j}^n &= (-1)^{2i+\frac{k}{2}+j+n}Y_{ij}^{\frac{k}{2}-n}
\end{aligned} \tag{B.3}$$

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All of these formulas can be easily derived from the general one, or from the symmetries of the SU(2) P matrix.
In addition to all this, there is a formula that relates the fusion coefficients to the Y tensor, namely

$$Y^{j_2}_{Jj_1} = Z^{j_2}_{Jj_1} \sum_{j=0}^{[\frac{k}{2}]} N^j_{JJ} N^j_{j_1 j_2} (-1)^j \tag{B.4}$$

where $Z^{j_2}_{Jj_1}$ is only a sign. Since the sum on the right hand side is invariant under $J \mapsto \frac{k}{2} - J$ or under $(j_1, j_2) \mapsto (\frac{k}{2} - j_1, \frac{k}{2} - j_2)$, we need only to give the values of $Z^{j_2}_{Jj_1}$ for $J \leq \frac{k}{4}$ and $j_1 \leq \frac{k}{4}$, because the rest can be found by using the symmetry relations for the Y tensor. One finds

$$Z^{j_2}_{Jj_1} = (-1)^{2J} \epsilon \qquad \text{for } J, j_1 \leq \frac{k}{4} \tag{B.5}$$

where ϵ is equal to one except when all of the following conditions are fulfilled: $j_2 > \frac{k}{4}$, $2J \geq \min\{j_1 + j_2, k - j_1 - j_2\}$, $2J + j_2 - j_1$ is odd and one of the following is true:

- k is odd and $j_1 \in \mathbb{N} + \frac{1}{2}$
- k is even, $j_1 \in \mathbb{N}$ and $j_1 + j_2 > \frac{k}{2}$.

Putting all together, we write

$$\begin{aligned} Z^{j_2}_{Jj_1} &= (-1)^{2J} \epsilon & \qquad Z^{j_2}_{\frac{k}{2}-J j_1} &= (-1)^{2J+2j_1+k} \epsilon \\ Z^{\frac{k}{2}-j_2}_{J \frac{k}{2}-j_1} &= (-1)^{j_1 - j_2} \epsilon & \qquad Z^{\frac{k}{2}-j_2}_{\frac{k}{2}-J \frac{k}{2}-j_1} &= (-1)^{2J+j_1+j_2+k} \epsilon \end{aligned} \tag{B.6}$$

with ϵ as above and $J, j_1 \leq \frac{k}{4}$.

C. P and Y matrices

For convenience, we list some orientifold specific data for the theories $\mathrm{SU}(2)_k$, $\mathrm{U}(1)_k$ and the coset model, all of which can be found in the appendix of [22].

$\mathrm{SU}(2)_k$. The P-matrix of the level k SU(2) WZW model is

$$P_{jj'} = \frac{2}{\sqrt{k+2}} \sin\left[\frac{\pi(2j+1)(2j'+1)}{2(k+2)}\right] \delta^{(2)}_{2j+2j'+k}.$$

The components of the Y tensor are explicitly evaluated in the separate appendix B.

$\mathrm{U}(1)_k$. The P-matrix and Y-tensor of the level k U(1) is

$$\begin{aligned} P_{nn'} &= \frac{1}{\sqrt{k}} \mathrm{e}^{-\frac{\pi i \widehat{n} \widehat{n'}}{2k}} \delta^{(2)}_{n+n'+k}, \\ Y^{n''}_{nn'} &= \delta^{(2)}_{n'+n''} \left(\delta^{(2k)}_{n+\frac{\widehat{n'}-\widehat{n''}}{2}} + (-1)^{n'+k} \delta^{(2k)}_{n+\frac{\widehat{n'}-\widehat{n''}}{2}+k} \right), \end{aligned}$$

where $\widehat{n}$ is the unique member of $n + 2k\mathbb{Z}$ in the standard range $\{-k+1, \ldots, k-1, k\}$. In the following, we will omit the $\hat{}$, but it is understood that all labels are chosen in this way.

Minimal model. We first note that the Q-matrix $Q = ST^2S$ of the minimal model can be expressed in terms of the Q-matrices of the constituent theories in the following way

$$Q_{(j,n,s)(j',n',s')} = Q_{jj'}Q^*_{nn'}Q_{ss'} + Q_{j(\frac{k}{2}-j')}Q^*_{n(n'\hat{+}(k+2))}Q_{s(s'\hat{+}2)} \tag{C.1}$$

The P-matrix is then obtained as

$$\begin{aligned} P_{(j,n,s)(j'n's')} &= T^{\frac{1}{2}}_{(j,n,s)}Q_{(j,n,s)(j',n',s')}T^{\frac{1}{2}}_{(j',n',s')} \\ &= \sigma_{j,n,s}\sigma_{j'n's'}P_{jj'}P^*_{nn'}P_{ss'} \\ &\qquad + \sigma_{j,n,s}\sigma_{\frac{k}{2}-j',n'\hat{+}(k+2),s'\hat{+}2}P_{j,\frac{k}{2}-j'}P^*_{n,n'\hat{+}(k+2)}P_{s,s'\hat{+}2} \\ &= \sigma_{j,n,s}\sigma_{j'n's'}\left(P_{jj'}P^*_{nn'}P_{ss'} + (-1)^{\frac{|n'|-|s'|-2j'}{2}}P_{j,\frac{k}{2}-j'}P^*_{n,n'\hat{+}(k+2)}P_{s,s'\hat{+}2}\right) \end{aligned} \tag{C.2}$$

An explicit expression for the P-matrix is

$$\begin{aligned} P_{(j,n,s)(j'n's')} = {}& \sigma_{j,n,s}\sigma_{j'n's'}\frac{\sqrt{2}}{k+2}\delta^{(2)}_{s+s'}\ \mathrm{e}^{\frac{\pi i n n'}{2(k+2)}}\ \mathrm{e}^{-\frac{\pi i s s'}{4}}\left(\sin\left[\pi\frac{(2j+1)(2j'+1)}{2(k+2)}\right]\delta^{(2)}_{2j+2j'+k}\ \delta^{(2)}_{n+n'+k}\right. \\ & \left. +(-1)^{\frac{2j'+n'+s'}{2}}\mathrm{e}^{\frac{\pi i(n+s)}{2}}\sin\left[\pi\frac{(2j+1)(k-2j'+1)}{2(k+2)}\right]\delta^{(2)}_{2j+2j'}\ \delta^{(2)}_{n+n'}\right) \end{aligned} \tag{C.3}$$

To decompose the Y matrix for the minimal model into Y matrices of the constituent theories, one first introduces the quantity

$$\tilde{Y}^c_{ab} = \sum_d \frac{S_{ab}Q_{bd}Q^*_{cd}}{S_{0d}} = \sqrt{\frac{T_c}{T_b}}\ Y^c_{ab}. \tag{C.4}$$

One then finds

$$\tilde{Y}^{(j'',n'',s'')}_{(j,n,s)(j',n',s')} = \tilde{Y}^{j''}_{jj'}\ \overline{\tilde{Y}}{}^{n''}_{nn'}\tilde{Y}^{s''}_{ss'} + \tilde{Y}^{\frac{k}{2}-j''}_{jj'}\ \overline{\tilde{Y}}{}^{n''+k+2}_{nn'}\ \tilde{Y}^{s''+2}_{ss'}. \tag{C.5}$$

This decomposition allows us to evaluate the following combination of Y-matrices

$$\begin{aligned} & Y^{(j'',n'',s'')}_{(j,n,s)(j',n',s')} + Y^{(j'',n'',s'')}_{(j,n,s+2)(j',n',s')} = 2\sigma_{j_1,n_1,s_1}\sigma_{j_2,n_2,s_2}\delta^{(2)}_{s'}\ \delta^{(2)}_{s''} \\ & \left(Y^{j''}_{jj'}Y^{n''}_{nn'}\ \delta^{(4)}_{2s+s'-s''} + (-1)^{\frac{|n|-|s|-2j}{2}}Y^{\frac{k}{2}-j''}_{jj'}\ Y^{n''\hat{+}(k+2)}_{nn'}\ \delta^{(4)}_{2s+s'-s''+2}\right) \end{aligned} \tag{C.6}$$

Here, we have made use of the explicit form of the $\mathrm{U}(1)_2$ Y matrix, from which one can derive the property

$$Y^{s''}_{ss'} + Y^{s''}_{s\hat{+}2\ s'} = 2\delta^{(2)}_{s'}\ \delta^{(2)}_{s''}\ \delta^{(4)}_{2s+s'-s''} \tag{C.7}$$

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